ES. 1

$$\underset{n \times p}{Z} \quad R = \begin{bmatrix} 1 & \cdots & r \\ r & \ddots & \vdots \\ \vdots & \ddots & r \\ r & \cdots & 1 \end{bmatrix} \quad r > 0$$

$$\lambda_1 = 1 + (p-1)r \qquad v_1 = \begin{bmatrix} \frac{1}{\sqrt{p}} \\ \vdots \\ \frac{1}{\sqrt{p}} \end{bmatrix}$$

1) $Y_1 = Z \underline{v}_1$, λ_1

2) Per quali r e p la var. spiegata da Y_1 è elevata? Es. numerico

<u>SOL</u>:

$$Y_1 = Z \underline{v}_1 = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1p} \\ \vdots & \vdots & & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{np} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{p}} \\ \vdots \\ \frac{1}{\sqrt{p}} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{\sqrt{p}} z_{11} + \cdots + \frac{1}{\sqrt{p}} z_{1p} \\ \vdots \\ \frac{1}{\sqrt{p}} z_{n1} + \cdots + \frac{1}{\sqrt{p}} z_{np} \end{bmatrix} = \frac{1}{\sqrt{p}} \begin{bmatrix} \sum_{j=1}^{p} z_{1j} \\ \vdots \\ \sum_{j=1}^{p} z_{nj} \end{bmatrix}$$

$$\lambda_1 = 1 + (p-1)r$$

% var spiegata da Y_1 $= \dfrac{\lambda_1}{Tr(R)} = \dfrac{1+(p-1)r}{p} =$

$$= \frac{1+rp-r}{p} = \frac{rp}{p} + \frac{1-r}{p} = r + \frac{1-r}{p}$$

$r \in [0,1]$

- $r \approx 1$: $\frac{\lambda_1}{Tr(R)} \approx 1 + 0 = 1$ ALTA!
- $r \approx 0$: $\frac{\lambda_1}{Tr(R)} \approx 0 + \frac{1}{p}$ ALTA se $p \approx 0$

$r = 0.8$
$p = 5$ $\Rightarrow \frac{\lambda_1}{Tr(R)} = 0.84$

ES 2:

H matr. centram. dati

a) $Tr(H)$ b) $\underset{n\times n}{H}\,\underset{n\times 1}{\underline{1}}$ c) $\underset{n\times n}{H}\,\underset{n\times 1}{\underline{a}}$ $\sum_{i=1}^{n} a_i = 0$

SOL:

$$H = \underset{n\times n}{I} - \frac{1}{n}\,\underset{n\times 1}{\underline{1}}\,\underset{1\times n}{\underline{1}^T}$$

a) $$Tr(H) = Tr\left(I - \frac{1}{n}\underline{1}\,\underline{1}^T\right) =$$
$$= Tr(I) - Tr\left(\frac{1}{n}\underline{1}\,\underline{1}^T\right) =$$
$$= Tr(I) - \frac{1}{n} Tr\left(\underline{1}\,\underline{1}^T\right) =$$
$$= n - \frac{1}{\cancel{n}}\cancel{n} = n - 1$$

b) $$H\underline{1} = \left(I - \frac{1}{n}\underline{1}\,\underline{1}^T\right)\underline{1} = I\underline{1} - \frac{1}{n}\underline{1}\left(\underline{1}^T\underline{1}\right) =$$
$$[1 \dots 1]\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} = n$$
$$= \underline{1} - \frac{1}{\cancel{n}}\underline{1}\,\cancel{n} = 0$$

c) $$H\underline{a} = \left(I - \frac{1}{n}\underline{1}\,\underline{1}^T\right)\underline{a} = I\underline{a} - \frac{1}{n}\underline{1}\left(\underline{1}^T\underline{a}\right) =$$

$$[1 \dots 1]\begin{bmatrix} \vdots \\ a_m \end{bmatrix} = \sum_{i=1} a_i = 0$$

$$\sum a_i = 0$$

$$= \underline{a} - \frac{1}{m}\underline{1}\underbrace{\sum_{i=1}^{m} a_i}_{=0} = \underline{a}$$

ES3:

SVD

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}$$

SOL:

$$\underset{m\times k}{A} = \underset{m\times m}{U}\ \underset{m\times k}{\Delta}\ \underset{k\times k}{V^T}$$

- U = autovett. $\underset{m\times m}{AA^T}$
- V = autovett. $\underset{k\times k}{A^TA}$
- $\Delta = \begin{bmatrix} \delta_1 & 0 & \dots & 0 \\ 0 & \delta_2 & \dots & 0 \\ \vdots & & \delta_{\min(m,k)} & 0 \\ 0 & \dots & & 0 \end{bmatrix}$

$\delta_j \geq 0 \ \forall j$

$\delta_1 \geq \dots \geq \delta_{\min(m,k)}$

$\sqrt{\text{autovalori} \geq 0 \text{ di } A^TA}$
(AA^T)

SOL:

1) $AA^T = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{pmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} = K$

$A^TA = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix} = H$

2) autovalori e autovett. di K

$\det(K - \lambda I) = 0$

$$\det\begin{pmatrix} 2-\lambda & 0 & 4 \\ 0 & 8-\lambda & 0 \\ 4 & 0 & 8-\lambda \end{pmatrix} = (8-\lambda)\det\begin{pmatrix} 2-\lambda & 4 \\ 4 & 8-\lambda \end{pmatrix} =$$

$$=(8-\lambda)\left((2-\lambda)(8-\lambda)-16\right)=$$
$$=(8-\lambda)\,\lambda\,(\lambda-10)=0$$
$$\lambda_1=0 \quad \vee \quad \lambda_2=8 \quad \vee \quad \lambda_3=10$$

autovettore relativo a $\lambda_2=8$

$$(K-\lambda_2 I)\,\underline{v}_2=\underline{0}$$

$$\begin{bmatrix} -6 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6v_{21}+4v_{23}=0 \\ 0=0 \\ 4v_{21}=0 \end{cases} \Rightarrow \begin{cases} v_{23}=0 \\ 0=0 \rightarrow v_{22}=1 \\ v_{21}=0 \end{cases}$$

voglio $\|\underline{v}_2\|=1$

$$\Updownarrow$$

$$\sqrt{v_{21}^2+v_{22}^2+v_{23}^2}=1$$

$$\Rightarrow \underline{v}_2=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{v}_1=\begin{bmatrix} -\frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\underline{v}_3=\begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}$$

3) autovalori e vettori di H

$$\det(H-\lambda I)=0$$

$$\det\begin{pmatrix} 9-\lambda & 1 \\ 1 & 9-\lambda \end{pmatrix}=0$$

$$\Updownarrow$$

$$(9-\lambda)^2-1=0$$

$$(\lambda-8)(\lambda-10)=0$$

$$\lambda_1 = 8 \lor \lambda_2 = 10$$

$$(H - \overset{8}{\lambda_1} I)\,\underline{w}_1 = \underline{0} \iff \underline{w}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$(H - \underset{10}{\lambda_2} I)\,\underline{w}_2 = 0 \iff \underline{w}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

4) SVD : $A = U \Delta V^T$

$$\Delta = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{8} \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} \underline{v}_3 & \underline{v}_2 & \underline{v}_1 \end{bmatrix} \qquad V = \begin{bmatrix} \underline{w}_2 & \underline{w}_1 \end{bmatrix}$$

$$\lambda_3 = 10 \quad \lambda_2 = 8 \quad \lambda_1 = 0 \qquad\qquad \lambda_2 = 10 \quad \lambda_1 = 8$$

$$A = U \Delta V^T =$$

$$= \begin{bmatrix} 1/\sqrt{5} & 0 & -\frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{8} \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T}_{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}} =$$