Correzione simulazione esame AE

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ES.1

$$Z_1 = 0.9 f + u_1$$

$$Z_2 = 0.7 f + u_2$$

$$Z_3 = 0.5 f + u_3$$

$$Var(f) = 1$$

$$\Psi = Cov(u) = \begin{bmatrix} 0.19 & & \\ & 0.51 & \\ & & 0.75 \end{bmatrix}$$

1) Σ

2) comunalità e $Cor(f, Z_i)$

SOL:

$$Z = \Lambda f + \varepsilon$$

$$\Lambda = \begin{bmatrix} 0.9 \\ 0.7 \\ 0.5 \end{bmatrix} \qquad \Sigma = \Lambda \Lambda^T + \Psi$$

$p \times k = 3 \times 1$

$$\Lambda^T = [0.9 \quad 0.7 \quad 0.5]$$

$$\Sigma = \begin{bmatrix} 0.9 \\ 0.7 \\ 0.5 \end{bmatrix} [0.9 \quad 0.7 \quad 0.5] + \Psi =$$

$$= \begin{bmatrix} 0.81 & 0.9 \cdot 0.7 & 0.9 \cdot 0.5 \\ 0.63 & 0.49 & \\ 0.45 & 0.35 & 0.25 \end{bmatrix} + \Psi =$$

h_2^2 h_3^2

$$= \begin{bmatrix} 0.81 + 0.19 & & \\ 0.63 & 0.49 + 0.51 & \\ 0.45 & 0.35 & 0.25 + 0.75 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ 0.63 & 1 & \\ 0.45 & 0.35 & 1 \end{bmatrix}$$

2) $Var(z_i) = h_i^2 + \Psi_i$

$$h_i^2 = \sum_{j=1}^{K} \lambda_{ij}^2$$

$h_1^2 = 0.81$ $h_2^2 = 0.49$

$h_3^2 = 0.25$

$$Cor(z_i, f) = \frac{cov(z_i, f_1)}{\underbrace{\sqrt{Var(z_i)}}_{1} \underbrace{\sqrt{Var(f_1)}}_{1}}$$

$Cor(z_1, f) = \lambda_1 = 0.9$

$Cor(z_2, f) = \lambda_2 = 0.7$

$$\text{Cor}(Z_3, f) = \lambda_3 = 0.5$$

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ES 2

$\underset{m \times m}{A}$ idempotente

Gli autovalori di A $\lambda_i \in \{0, 1\}$ $\forall i = 1, \dots, m$

Sol:

λ_i = autovalore di A

v_i = autovettore di A

$$A \underline{v}_i = \lambda_i \underline{v}_i$$

$$AA = A$$

$$\underbrace{AA}_{A} \underline{v}_i = A \lambda_i \underline{v}_i$$

$$\underbrace{A \underline{v}_i}_{\lambda_i \underline{v}_i} = \lambda_i \underbrace{A \underline{v}_i}_{\lambda_i \underline{v}_i}$$

$$\lambda_i \underline{v}_i = \lambda_i^2 \underline{v}_i$$

$$(\lambda_i - \lambda_i^2) \underbrace{\underline{v}_i}_{\neq 0} = 0 \Uparrow$$

$$= \Downarrow$$

$$\lambda_i - \lambda_i^2 = 0$$

$$\lambda_i (1 - \lambda_i) = 0$$

$$\lambda_i = 0 \lor \lambda_i = 1$$

ES 3

$\underset{n \times p}{X}$ $\quad S = \text{diag}(s_{11}, \dots, s_{pp})$

$s_{11} \geq \dots \geq s_{pp} > 0$

1) ? = PCA su S e R

2) ? = AF su R

SOL:

1) Svolgo PCA su X e mostro che Y (PC) $Y = X$

$$\boxed{Y = \tilde{X} V}$$

autovettori di S

$\to \tilde{X} = X$

$\to V = I \Rightarrow Y = \tilde{X}$

(1) autovalori

$$\det(S - \lambda I) = 0$$

$$\parallel$$

$$\det \begin{bmatrix} s_{11}-\lambda & & 0 \\ & \ddots & \\ 0 & & s_{pp}-\lambda \end{bmatrix} =$$

$$= \prod_{i=1}^{p} (s_{ii}-\lambda) = 0$$

$$\Updownarrow$$

$$\lambda_1 = s_{11} \vee \dots \vee \lambda_p = s_{pp}$$

(2) $\boxed{S \underline{v}_j = \lambda_j \underline{v}_j}$ j fissato

$$\begin{bmatrix} s_{11} & & 0 \\ & \ddots & \\ 0 & & s_{pp} \end{bmatrix} \begin{bmatrix} v_{j1} \\ \vdots \\ v_{jp} \end{bmatrix} = \begin{bmatrix} s_{11} v_{j1} \\ \vdots \\ s_{pp} v_{jp} \end{bmatrix}$$

$$[s_{11}\ 0 \dots 0] \begin{bmatrix} v_{j1} \\ \vdots \\ v_{jp} \end{bmatrix} = s_{11} v_{j1}$$

$$\begin{bmatrix} s_{11} v_{j1} \\ \vdots \\ s_{pp} v_{jp} \end{bmatrix} = s_{jj} \begin{bmatrix} v_{j1} \\ \vdots \\ v_{jp} \end{bmatrix} \leftarrow S \underline{v}_j = \lambda_j \underline{v}_j$$

$\lambda_j = s_{jj}$

$$\begin{cases} s_{11} v_{j1} = s_{jj} v_{j1} \\ \vdots \\ s_{pp} v_{jp} = s_{jj} v_{jp} \end{cases} \qquad \begin{cases} \overbrace{(s_{11} - s_{jj})}^{\neq 0} v_{j1} = 0 \\ \vdots \\ \underbrace{(s_{pp} - s_{jj})}_{\neq 0} v_{j1} = 0 \end{cases}$$

$$s_{ii} - s_{jj} \neq 0 \quad \forall i \neq j$$

$$\Rightarrow (s_{ii} - s_{ii}) v_{ii} = 0$$

$$\Updownarrow$$

$$v_{ji} = 0 \quad \forall\, i \neq j$$

Se $i = j$ $\quad \underbrace{(s_{jj} - s_{jj})}_{0} \underset{\substack{\shortparallel \\ 1}}{v_{jj}} = 0$

$$S\underline{v}_j = \lambda_j \underline{v}_j \iff \underline{v}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j$$

$$V = [\underline{v}_1 \; \dots \; \underline{v}_p] = \underline{I}$$

$$\Downarrow$$

$$Y = \tilde{X} V = \tilde{X} \underline{I} = \tilde{X}$$

non ha senso effettuare PCA su S e per lo stesso ragionamento non ha senso nemmeno su R

2) $\underline{\Lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$ $\quad \exists\, 1!$ fattore

$$S = \underline{\Lambda}\,\underline{\Lambda}^T + \underline{\Psi} =$$

$$= \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} + \underline{\Psi} =$$

$$= \begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 \end{bmatrix} + \underline{\Psi} =$$

$$= \begin{bmatrix} \lambda_1^2+\psi_1 & & \\ 0 = \lambda_1\lambda_2 & \lambda_2^2+\psi_2 & \\ 0 = \lambda_1\lambda_3 & \underbrace{\lambda_2\lambda_3}_{=0} & \lambda_3^2+\psi_3 \end{bmatrix}$$

$$\begin{cases} \lambda_1\lambda_2 = 0 \\ \lambda_1\lambda_3 = 0 \\ \lambda_2\lambda_3 = 0 \end{cases} \qquad \begin{cases} \lambda_1 = 0 \lor \lambda_2 = 0 \\ \lambda_1 = 0 \lor \lambda_3 = 0 \\ \lambda_2 = 0 \lor \lambda_3 = 0 \end{cases}$$

Se scelgo $\lambda_1 = 0$

$\Rightarrow$ $\lambda_1\lambda_2 = 0$

$\lambda_1\lambda_3 = 0$

affinché $\lambda_2\lambda_3 = 0$ devo richiedere $\lambda_2 = 0 \lor \lambda_3 = 0$

Scelgo $\lambda_2 = 0$

$$\underline{X} = \underline{\Lambda}\, f + \underline{\varepsilon} =$$

$$= \begin{bmatrix} 0 \\ 0 \\ \lambda_3 \end{bmatrix} f + \underline{\varepsilon}$$

$$\underline{X} = \underline{\Lambda}\, f + \underline{\varepsilon} \Leftrightarrow \begin{cases} X_1 = \varepsilon_1 \\ X_2 = \varepsilon_2 \\ X_3 = \lambda_3 f + \varepsilon_3 \end{cases}$$