Stability Selection

Statistical Learning
CLAMSES - University of Milano-Bicocca

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References

- Meinshausen, Buhlmann (2010). Stability selection. JRSS-B, 72:417–473
- Shah, Samworth (2013). Variable selection with error control: another look at stability selection. JRSS-B, 75:55–80.

Stability path

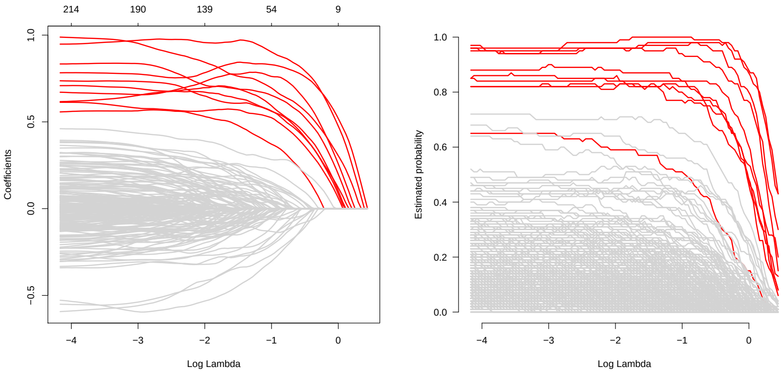
- The *regularisation path* of the lasso is

$$\{\hat{\beta}_j(\lambda), j = 1, \ldots, p, \lambda \in \Lambda\}$$

- The *stability path* is

$$\{\hat{\pi}_j(\lambda), j = 1, \ldots, p, \lambda \in \Lambda\}$$

  where $\hat{\pi}_j(\lambda)$ is the estimated probability for the *j*th predictor to be selected by the lasso(λ) when randomly resampling from the data



214
190
139
54
9
Coefficients
1.0
0.5
0.0
−0.5
−4
−3
−2
−1
0
Log Lambda
Estimated probability
1.0
0.8
0.6
0.4
0.2
0.0
−4
−3
−2
−1
0
Log Lambda

Algorithm 1 Stability Path Algorithm with the Lasso

Require: $B \in \mathbb{N}$, Λ grid, $\tau \in (0.5, 1)$

for $b = 1, \ldots, B$ **do**

  Randomly select $n/2$ indices from $\{1, \ldots, n\}$;

  Perform the lasso on the $n/2$ observations to obtain

$$\hat{S}_{n/2}(\lambda) = \{j : \hat{\beta}_j(\lambda) \neq 0\} \qquad \forall\ \lambda \in \Lambda$$

end for

Compute the relative selection frequencies:

$$\hat{\pi}_j(\lambda) = \frac{1}{B} \sum_{b=1}^{B} \mathbb{1}\{j \in \hat{S}_{n/2}(\lambda)\} \qquad \forall\ \lambda \in \Lambda$$

The set of *stable predictors* is given by

$$\hat{S}_{\text{stab}} = \{j : \max_{\lambda \in \Lambda} \hat{\pi}_j(\lambda) \geq \tau\}$$

Algorithm 2 (Complementary Pairs) Stability Selection

Require: A variable selection procedure $\hat{S}_n$, $B \in \mathbb{N}$, $\tau \in (0.5, 1)$

for $b = 1, \dots, B$ **do**

  Split $\{1, \dots, n\}$ into (I^{2b-1}, I^{2b}) of size $n/2$, and for each get

$$\hat{S}_{n/2}^{2b-1} \subseteq \{1, \dots, p\}, \qquad \hat{S}_{n/2}^{2b} \subseteq \{1, \dots, p\}$$

end for

Compute the relative selection frequencies:

$$\hat{\pi}_j = \frac{1}{2B} \sum_{b=1}^{B} \left(\mathbb{1}\{j \in \hat{S}_{n/2}^{2b-1}\} + \mathbb{1}\{j \in \hat{S}_{n/2}^{2b}\} \right)$$

The set of *stable predictors* is given by

$$\hat{S}_{\text{stab}} = \{j : \hat{\pi}_j \geq \tau\}$$

- The relative selection frequency $\hat{\pi}_j$ is an unbiased estimator of

$$\pi_j^{n/2} = \mathrm{P}(j \in \hat{S}_{n/2})$$

  but, in general, a biased estimator of

$$\pi_j^n = \mathrm{P}(j \in \hat{S}_n) = \mathbb{E}(\mathbb{1}\{j \in \hat{S}_n\})$$

- The key idea of stability selection is to improve on the simple estimator $\mathbb{1}\{j \in \hat{S}_n\}$ of π_j^n through subsampling.
- By means of averaging involved in $\hat{S}_{\mathrm{stab}}$, we hope that $\hat{\pi}_j$ will have reduced variance compared to $\mathbb{1}\{j \in \hat{S}_n\}$ and this increased stability will more than compensate for the bias incurred.

Theorem

Assume that

1. $\{\mathbb{1}\{j \in \hat{S}_{n/2}\}, j \in N\}$ *is exchangeable;*
2. *The variable selection procedure is not worse than random guessing, i.e.*

$$\frac{\mathbb{E}(|\hat{S}_{n/2} \cap S|)}{\mathbb{E}(|\hat{S}_{n/2} \cap N|)} \geq \frac{|S|}{|N|}.$$

Then, for $\tau \in (1/2, 1]$

$$\mathbb{E}(|\hat{S}_{\text{stab}} \cap N|) \leq \frac{1}{(2\tau - 1)} \frac{q^2}{p}$$

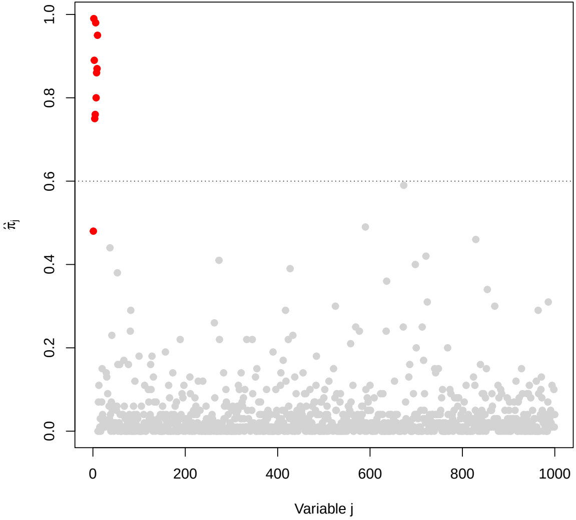
where $q = \mathbb{E}(|\hat{S}_{n/2}|)$

- The choice of the number of subsamples B is of minor importance
- It is possible to fix $q = \mathbb{E}(|\hat{S}_{n/2}|)$ and run the variable selection procedure until it selects q variables. However, if q is too small, one would select only a subset of the signal variables as

$$|\hat{S}_{\text{stab}}| \leq |\hat{S}_{n/2}| = q$$

- For example, with $p = 1000$, $q = 50$ and $\tau = 0.6$ then

$$\mathbb{E}(|\hat{S}_{\text{stab}} \cap N|) \leq 12.5$$



1.0
0.8
0.6
0.4
0.2
0.0
$\hat{\pi}_j$
0
200
400
600
800
1000
Variable j